DAMPED HARMONIC OSCILLATORS
IN THE HOLOMORPHIC REPRESENTATION

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Abstract

Quantum dynamical semigroups are applied to the study of the time evolution of harmonic oscillators, both bosonic and fermionic. Explicit expressions for the density matrices describing the states of these systems are derived using the holomorphic representation. Bosonic and fermionic degrees of freedom are then put together to form a supersymmetric oscillator; the conditions that assure supersymmetry invariance of the corresponding dynamical equations are explicitly derived.

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1. INTRODUCTION

The dynamics of a small system $\mathcal{S}$ in interaction with a large environment $\mathcal{E}$ is in general very complex and can not be described in terms of evolution equations that are local in time. Possible initial correlations and the continuous exchange of energy as well as entropy between the $\mathcal{S}$ and $\mathcal{E}$ produce phenomena of irreversibility and dissipation.

Nevertheless, there are instances for which simple and mathematically precise description of the subdynamics can actually be given. When the typical time scale in the evolution of the subsystem $\mathcal{S}$ is much larger than the characteristic time correlations in the environment, one expects (and actually proves) the disappearance of memory and non-linear phenomena, although quantum coherence is usually lost.[1-5]

In such cases, the states of $\mathcal{S}$, conveniently described by a density matrix ρ, are seen to evolve in time by means of a family of linear maps that obey very basic physical requirements, like forward in time composition (semigroup property) and complete positivity. They form a so-called quantum dynamical semigroup.[1-3]

This description of the time evolution of open quantum systems is actually very general; it is applicable to all physical situations for which the interaction between $\mathcal{S}$ and $\mathcal{E}$ can be considered to be weak and for times for which non-linear disturbances due to possible initial correlations have disappeared. In particular, quantum dynamical semigroups have been used to model laser dynamics in quantum optics,[6-8] to study the evolution of various statistical systems,[1-3] to analyze the interaction of a microsystem with a macroscopic apparatus.[9-11]

Recently, they have been used to describe effects leading to irreversibility and dissipation in elementary particle physics phenomena. Non-standard low energy effects accompanied by loss of quantum coherence are in fact expected to appear as a consequence of gravitational quantum fluctuations at Planck's scale.[12] Detailed analysis of these effects have been performed in the system of neutral mesons,[13-16] in neutron interferometry,[17] neutrino oscillations,[18] and in the propagation of polarized photons;[19, 20] the outcome of these investigations is that present and future elementary particle experiments will likely put stringent bounds on these non-standard dissipative phenomena.

These studies, in particular those dealing with correlated neutral mesons,[14, 21] have also further clarified the importance of the condition of complete positivity in the description of open quantum systems. In many investigations complete positivity is often replaced by the milder condition of simple positivity; this guarantees the positivity of the eigenvalues of the density matrix of the subsystem $\mathcal{S}$, but not that of a more general system obtained by trivially coupling $\mathcal{S}$ with an arbitrary n-level system. Lack of imposing this more stringent requirement could lead to unacceptable physical consequences, like the appearance of negative probabilities.[21]

To further analyze the properties of the quantum dynamical semigroup description of open systems, we shall apply this general framework to the analysis of the evolution of one dimensional oscillators, both bosonic and fermionic (for earlier investigations on the bosonic case, see [22-25] and references therein). We shall adopt the holomorphic representation [26, 27, 23] since it allows an explicit description of the relevant density matrices in terms of complex and Grassmannian (anticommuting) variables; in the most simple situations, the general form of these density matrices turns out to be Gaussian. This

allows the explicit evaluation of the corresponding (von Neumann) entropy and analysis of its time evolution. Finally, we shall combine a bosonic and a fermionic degrees of freedom to form a supersymmetric oscillator. We shall then derive the conditions that guarantee the supersymmetric invariance of the dynamical equations and discuss how these affect the time evolution of the total density matrix.

2. THE BOSONIC OSCILLATOR

As explained in the introductory remarks, we shall study the dynamics of a single oscillator in interaction with a large environment. The states of the system will be represented by a density matrix ρ_B, *i.e.* by a positive hermitian operator, with constant trace, acting on the bosonic Hilbert space $\mathcal{H}_B$. Our analysis is based on the assumption that its time evolution is given by a quantum dynamical semigroup; this is a completely positive, trace preserving, one parameter (=time) family of linear maps, acting on the set $\{\rho\}$ of bosonic density matrices. These maps are generated by equations of the following general form:[1-3]

$$\frac{\partial \rho(t)}{\partial t} = \mathcal{L}[\rho(t)] \equiv -i\big[H, \rho(t)\big] + L[\rho(t)] \ . \tag{2.1}$$

The first term in $\mathcal{L}$ is the standard quantum mechanical one, that contains the system hamiltonian H, driving the time evolution in absence of the environment. In the case of the bosonic oscillator, it can be taken to have the most general quadratic form in the bosonic creation $a^\dagger$ and annihilation a operators:

$$H_B = \frac{1}{2}\big[\omega_B\,(a^\dagger a + a a^\dagger) + \mu\,a^2 + \mu^*\,a^{\dagger 2}\big] \ , \tag{2.2}$$

where $\omega_B \geq 0$ and μ is a complex parameter (the star means complex conjugation). The second piece $L[\rho]$ takes into account the interaction with the environment; it is a linear map, whose form is fully determined by the requirement of complete positivity and trace conservation:

$$L[\rho] = -\frac{1}{2}\sum_k \left(L_k^\dagger L_k\,\rho + \rho\,L_k^\dagger L_k\right) + \sum_k L_k\,\rho\,L_k^\dagger \ . \tag{2.3}$$

The operators L_k should be chosen such that the expression in (2.3) is well defined. In absence of the term $L[\rho]$, pure states would be transformed into pure states. Instead, the additional piece (2.3) produce in general dissipation and loss of quantum coherence.

The choice of the operators L_k is largely arbitrary. However, since the hamiltonian H_B is quadratic in $a^\dagger$ and a, one is led to assume the same property also for the additional term $L[\rho]$. This implies a linear expression for the operators L_k:

$$L_k = r_k\,a + s_k\,a^\dagger \ , \tag{2.4}$$

with r_k and s_k complex parameters; this requirement further guarantees the exact solvability of the equation in (2.1). Note that the operators (2.4) are not bounded; nevertheless,

by adapting the arguments presented in Ref.[22] to the present case, one can show that the exponential map generated by (2.1) is well defined.

This description of the damped bosonic oscillator can be further simplified by means of a suitable canonical transformation. First, notice that not all values of the oscillator frequency ω_B and the complex parameter μ are physically allowed. Indeed, the spectrum of the hamiltonian in (2.2) is bounded from below only for:

$$\omega_B^2 - |\mu|^2 \geq 0 \ . \tag{2.5}$$

This is a consequence of the fact that H is an element of the Lie algebra $su(1,1)$, whose generators in the so-called metaplectic representation take the form:

$$K_0 = \frac{1}{4}(a^\dagger a + a a^\dagger) \ , \qquad (K_+)^\dagger = K_- = \frac{a^2}{2} \ . \tag{2.6}$$

The condition (2.5) guarantees that H can be unitarily "rotated" to an element of the Cartan algebra with spectrum bounded from below.[23]

In other terms, by means of a unitary canonical transformation, one can now pass to new operators:[26, 28]

$$\tilde{a} = \Phi \, a + \Psi \, a^\dagger \ ,$$
$$\tilde{a}^\dagger = \Psi^* \, a + \Phi^* \, a^\dagger \ , \tag{2.7}$$

with

$$\Phi = \sqrt{\frac{\omega_B + \Omega_B}{2 \, \Omega_B}} \ , \qquad \Psi = \frac{\mu^*}{\sqrt{2 \, \Omega_B(\omega_B + \Omega_B)}} \ , \qquad \Omega_B = \sqrt{\omega_B^2 - |\mu|^2} \ , \tag{2.8}$$

such that the hamiltonian take the simplified form:

$$H_B = \frac{\Omega_B}{2} \, \{\tilde{a}^\dagger, \tilde{a}\} \ . \tag{2.9}$$

The operators L_k in (2.3) are still linear in the new variables $\tilde{a}^\dagger$ and $\tilde{a}$, although with redefined coefficients.

This discussion explicitly shows that, without loss of generality, one can set $\mu = 0$ in (2.2); a non vanishing μ can always be reinstated at the end by undoing the transformation (2.7). With this choice, the evolution equation (2.1) for the bosonic oscillator becomes:

$$\frac{\partial \rho_B(t)}{\partial t} = \mathcal{L}_B[\rho_B(t)] \equiv -i\omega_B \big[a^\dagger a, \rho_B(t)\big] + L_B[\rho_B(t)] \ , \tag{2.10}$$

where, by inserting (2.4) into (2.3), one has:

$$L_B[\rho] = \eta_B \Big(\big[a\rho, a^\dagger\big] + \big[a, \rho a^\dagger\big] \Big) + \sigma_B \Big(\big[a^\dagger \rho, a\big] + \big[a^\dagger, \rho a\big] \Big)$$
$$- \lambda_B^* \big[a, [a, \rho]\big] - \lambda_B \big[a^\dagger, [a^\dagger, \rho]\big] \ , \tag{2.11}$$

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with

$$\eta_B = \frac{1}{2} \sum_k |r_k|^2 \ , \qquad \sigma_B = \frac{1}{2} \sum_k |s_k|^2 \ , \qquad \lambda_B = \frac{1}{2} \sum_k r_k^* s_k \ . \tag{2.12}$$

Note that from these expressions one deduces that:

$$\eta_B \geq 0 \ , \qquad \sigma_B \geq 0 \ , \qquad |\lambda_B|^2 \leq \eta_B \sigma_B \ , \tag{2.13}$$

the last relation being a consequence of the Schwartz inequality; let us remark that these are precisely the conditions that assure complete positivity of the time evolution generated by the operator L_B in (2.11).

In order to study the solutions of the equation (2.10), we shall work in the holomorphic representation;[26-28, 23] it allows deriving explicit expressions for the density matrix $\rho_B(t)$ so that its behaviour in various regimes can be more easily discussed. In this formulation, the elements $|\psi\rangle$ of the bosonic Hilbert space $\mathcal{H}_B$ are represented by holomorphic functions $\psi(\bar{z})$ of the complex variable $\bar{z}$, with inner product:[†]

$$\langle \phi | \psi \rangle = \int \psi^*(z) \, \phi(\bar{z}) \, e^{-\bar{z}z} \, d\bar{z} \, dz \ . \tag{2.14}$$

To every operator $\mathcal{O}$ acting on $\mathcal{H}_B$ there correspond a kernel $\mathcal{O}(\bar{z}, z)$ of two independent complex variables $\bar{z}$ and z, such that for the state $|\phi\rangle = \mathcal{O} \, |\psi\rangle$ one finds the representation:

$$\phi(\bar{z}) = \int \mathcal{O}(\bar{z}, w) \, \psi(\bar{w}) \, e^{-\bar{w}w} \, d\bar{w} \, dw \ . \tag{2.15}$$

In particular, the creation and annihilation operators, when acting on a state $|\psi\rangle$, are realized by multiplication and differentiation by the variable $\bar{z}$:

$$a^\dagger |\psi\rangle \to \bar{z} \, \psi(\bar{z}) \ , \qquad a |\psi\rangle \to \frac{\partial}{\partial \bar{z}} \psi(\bar{z}) \ , \tag{2.16}$$

while the identity operator is represented by $e^{\bar{z}z}$.

Since the term $\mathcal{L}_B$ in (2.10) is at most quadratic in $a^\dagger$ and a, the kernel $\rho_B(\bar{z}, z; t)$ representing the solution of (2.10) can be taken to be of generic Gaussian form:

$$\rho_B(\bar{z}, z; t) = \frac{1}{\sqrt{N(t)}} e^{-\frac{1}{2N(t)} \left[2 \, y(t) \, \bar{z}z - \bar{x}(t) \, z^2 - x(t) \bar{z}^2 \right] + \bar{z}z} \ . \tag{2.17}$$

Trace conservation for all times,

$$\text{Tr}[\rho_B(t)] = \int \rho_B(\bar{z}, z; t) \, e^{-\bar{z}z} \, d\bar{z} \, dz = 1 \ , \tag{2.18}$$

readily implies:

$$N(t) = y^2(t) - |x(t)|^2 \ , \tag{2.19}$$

[†] Here and in the following we use the conventions of Ref.[26]

while using (2.16) one finds that the unknown functions $x(t)$, $\bar{x}(t) = [x(t)]^*$ and $y(t)$ have the following physical meaning:

$$\langle a^2 \rangle(t) \equiv \text{Tr}\left[a^2\, \rho_B(t)\right] = \int \frac{\partial^2}{\partial \bar{z}^2} \left[\rho_B(\bar{z}, z; t)\right] e^{-\bar{z}z}\, d\bar{z}\, dz = x(t)\ ,$$

$$\langle a^{\dagger 2} \rangle(t) \equiv \text{Tr}\left[a^{\dagger 2}\, \rho_B(t)\right] = \int \bar{z}^2\, \rho_B(\bar{z}, z; t)\, e^{-\bar{z}z}\, d\bar{z}\, dz = \bar{x}(t)\ , \tag{2.20}$$

$$\langle a\, a^{\dagger} \rangle(t) \equiv \text{Tr}\left[a\, a^{\dagger}\, \rho_B(t)\right] = \int \frac{\partial}{\partial \bar{z}} \left[\bar{z}\, \rho_B(\bar{z}, z; t)\right] e^{-\bar{z}z}\, d\bar{z}\, dz = y(t)\ .$$

For simplicity, in writing (2.17) we have assumed $\langle a^{\dagger} \rangle(t) = \langle a \rangle(t) = 0$ for all times. As shown in the Appendix, this condition can be easily released starting with a more general Ansatz for $\rho_B(\bar{z}, z; t)$; it will not be needed for the considerations that follow.

Inserting (2.17) in the evolution equation (2.10), with the help of the relations (2.16) one finds that the unknown functions $x(t)$ and $y(t)$ satisfy the following linear equations:

$$\dot{x}(t) = -2\left(\eta_B - \sigma_B + i\omega_B\right)x(t) - 2\lambda_B\ ,$$
$$\dot{y}(t) = -2\left(\eta_B - \sigma_B\right)y(t) + 2\eta_B\ . \tag{2.21}$$

General solutions can be easily obtained. For initial values $x_0 \equiv x(0)$, $y_0 \equiv y(0)$ and $\eta_B \neq \sigma_B$, one finds:

$$x(t) = E(t)\, e^{-2i\omega_B t}\left(x_0 - x_\infty\right) + x_\infty\ ,$$
$$y(t) = E(t)\left(y_0 - y_\infty\right) + y_\infty\ , \tag{2.22}$$

where

$$E(t) = e^{-2(\eta_B - \sigma_B)t}\ , \qquad x_\infty = \frac{\lambda_B(\sigma_B - \eta_B + i\omega_B)}{(\eta_B - \sigma_B)^2 + \omega_B^2}\ , \qquad y_\infty = \frac{\eta_B}{\eta_B - \sigma_B}\ , \tag{2.23}$$

while in the particular case $\eta_B = \sigma_B$:

$$x(t) = e^{-2i\omega_B t}\left(x_0 - x_c\right) + x_c\ , \qquad x_c = i\frac{\lambda_B}{\omega_B}\ ,$$
$$y(t) = 2\eta_B t + y_0\ . \tag{2.24}$$

The large time behaviour of these solutions depends on the relative magnitude of the two positive parameters η_B and σ_B. Only when $\eta_B > \sigma_B$, the functions $x(t)$ and $y(t)$ have a well-defined limit. In this case, independently from the initial conditions, the density matrix $\rho_B(t)$ approaches for large t the equilibrium state ρ_B^∞, obtained substituting in (2.17) the asymptotic values x_∞ and y_∞ for $x(t)$ and $y(t)$. Indeed, ρ_B^∞ is clearly a fixed point of the evolution equation (2.10), for any value of η_B and σ_B.

Notice that ρ_B^∞ does not correspond in general to a thermal equilibrium state; to obtain an asymptotic Gibbs distribution, one has to set $\lambda_B = 0$ and introduce the inverse temperature β via the condition $y_\infty = [\coth(\beta\omega_B/2) + 1]/2$ (compare with (2.27) and (2.28) below), or equivalently $e^{\beta\omega_B} = \eta_B/\sigma_B$.

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On the other hand, when $\eta_B < \sigma_B$, the exponential term $E(t)$ in (2.22) blows up for large times, while in the special case $\eta_B = \sigma_B$, $y(t)$ grows linearly in time and $x(t)$ has an oscillatory behaviour. In both cases, for generic initial conditions, the normalization factor $N(t)$ in (2.19) grows unbounded, so that the functional $\rho_B(\bar{z}, z; t)$ becomes vanishingly small, while retaining its normalization, $\mathrm{Tr}[\rho_B(t)] = 1.^{\dagger}$

This peculiar behaviour can also be analyzed with the help of the Weyl operators:

$$W[\nu] = e^{\nu a + \bar{\nu} a^{\dagger}} \ . \tag{2.25}$$

By studying the time evolution of W induced by (2.10) via the relation $\mathrm{Tr}[W(t)\,\rho_B] \equiv \mathrm{Tr}[W\,\rho_B(t)]$, one finds that when $\eta_B > \sigma_B$ all Weyl operators remain well-defined for all t, approaching the identity for large times; on the other hand, for $\eta_B \leq \sigma_B$ one discovers that all Weyl operators vanish in the large time limit, except the identity $W[0]$, which is clearly a fixed point of the time evolution.

As mentioned in the Introduction, the entropy of an open system usually varies with time, due to the interaction with the environment. In many physical instances, monotonic increase of the von Neumann entropy,

$$S[\rho] \equiv -\mathrm{Tr}[\rho \ln \rho] = -\langle \ln \rho \rangle \ , \tag{2.26}$$

is a desirable property.[1-3, 13-20] For the damped oscillator described by the evolution equation (2.10), this request can not be fulfilled in general, if one insists on the existence of a well-behaved large-time equilibrium limit.

The explicit evaluation of $S_B \equiv S[\rho_B]$ is simplified by noticing that to the representation kernel (2.17) there corresponds the operator Ansatz:

$$\rho_B = \left[\frac{4}{\coth^2(\Omega/2) - 1} \right]^{1/2} e^{-\frac{1}{2}\left[A(aa^{\dagger} + a^{\dagger}a) + Ba^2 + \bar{B}a^{\dagger 2} \right]} \ , \tag{2.27}$$

where the parameters A and B are related to x and y of (2.17) through the relations:

$$x = -\frac{B}{2\Omega} \coth \frac{\Omega}{2} \ , \qquad y = \frac{1}{2}\left[\frac{A}{\Omega} \coth \frac{\Omega}{2} + 1 \right] \ , \qquad \Omega = \left(A^2 - |B|^2 \right)^{1/2} \ . \tag{2.28}$$

Inserting these relations in the definition (2.26), one obtains:

$$S_B = -\frac{1}{2} \ln \left[\frac{4}{\coth^2(\Omega/2) - 1} \right] + \frac{1}{2}\left[A \langle aa^{\dagger} + a^{\dagger}a \rangle + B \langle a^2 \rangle + \bar{B} \langle a^{\dagger 2} \rangle \right] \ . \tag{2.29}$$

It is now convenient to introduce the following quantity:

$$\chi^2 \equiv \frac{1}{4} \langle aa^{\dagger} + a^{\dagger}a \rangle^2 - \langle a^2 \rangle \langle a^{\dagger 2} \rangle$$
$$= (y - 1/2)^2 - |x|^2 = \frac{1}{4} \coth^2 \frac{\Omega}{2} \ , \tag{2.30}$$

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† For $\eta_B < \sigma_B$, the exception is given by $\rho_B(t) \equiv \rho_B^{\infty}$, since, as noted before, x_{∞} and y_{∞} are fixed points of (2.21). Note that y_{∞} blows up for $\eta_B = \sigma_B$, so that also ρ_B^{∞} becomes vanishingly small in this limit. As a consequence, the infinite temperature limit is singular.

such that $\chi \geq 1/2$; in terms of this variable, one can easily rewrite (2.29) as:

$$S_B = \left(\chi + \frac{1}{2}\right) \ln \left(\chi + \frac{1}{2}\right) - \left(\chi - \frac{1}{2}\right) \ln \left(\chi - \frac{1}{2}\right) . \tag{2.31}$$

The entropy S_B always grows with χ, starting at the minimum $S_B = 0$ for $\chi = 1/2$ and increasing as $\ln \chi$ for large χ. Recalling the explicit time-dependence of x and y in (2.22) and (2.24), one realizes that in general for $\eta_B \neq \sigma_B$ the variable χ does not monotonically grows with t, so that the condition $\dot{S}_B \geq 0$ can not be satisfied for all times. In particular, when $\eta_B > \sigma_B$ the equilibrium state ρ_B^∞ is reached in general at the expense of some negative entropy-exchange with the environment.

The case $\eta_B = \sigma_B$ is again special; in fact, the operators L_k in (2.4) are now hermitian and therefore the condition $\dot{S}_B \geq 0$ is guaranteed. Indeed, in this case χ approaches infinity for large times, and therefore so does S_B. Alternatively, using (2.3) in the definition (2.26), one can directly show that:[29]

$$\dot{S}_B \geq \left\langle \sum_k \left[L_k, L_k^\dagger\right] \right\rangle = 2(\sigma_B - \eta_B) . \tag{2.32}$$

Note however that $L_k = L_k^\dagger$ is only a sufficient condition for entropy increase. Indeed, for $\eta_B > \sigma_B$ take $x_0 = x_\infty$ and $y_0 \leq y_\infty$; in this case χ grows with time since $\dot{y}(t)$ is always positive, and therefore also S_B never decreases.

3. THE FERMIONIC OSCILLATOR

We shall now extend the analysis of the previous section to the case of a fermionic oscillator. The corresponding creation $\alpha^\dagger$ and annihilation α operators obey now the algebra:

$$\{\alpha, \alpha^\dagger\} = 1 , \qquad \alpha^2 = \alpha^{\dagger 2} = 0 . \tag{3.1}$$

As in the bosonic case, we shall assume the system in interaction with a large environment, and describe its time evolution by means of a quantum dynamical semigroup.

The states of the system will be described by an appropriate density matrix ρ_F, acting on the elements of the fermionic Hilbert space $\mathcal{H}_F$. This operator obeys an evolution equation of the form (2.1), where now the hamiltonian can be taken to be:

$$H_F = \frac{\omega_F}{2} [\alpha^\dagger, \alpha] , \qquad \omega_F \geq 0 . \tag{3.2}$$

Since $\alpha^\dagger$ and α are now nilpotent, the additional piece $L[\rho]$ in (2.3) turns out to be at most quadratic in these variables, and the operators L_k assume the generic form

$$L_k = r_k' \alpha + s_k' \alpha^\dagger . \tag{3.3}$$

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Inserting this in (2.3), one explicitly finds:

$$L_F[\rho] = \eta_F \left(2\, \alpha \rho \alpha^\dagger - \alpha^\dagger \alpha \rho - \rho \alpha^\dagger \alpha \right) + \sigma_F \left(2\, \alpha^\dagger \rho \alpha - \alpha \alpha^\dagger \rho - \rho \alpha \alpha^\dagger \right)$$
$$+ 2 \left(\lambda_F^* \, \alpha \rho \alpha + \lambda_F \, \alpha^\dagger \rho \alpha^\dagger \right) , \tag{3.4}$$

where the parameters η_F, σ_F and λ_F are as in (2.12) with the coefficients r_k and s_k replaced by the primed ones. Then, the complete evolution equation for the density matrix ρ_F takes the form:

$$\frac{\partial \rho_F(t)}{\partial t} = \mathcal{L}_F[\rho_F(t)] \equiv -i\omega_F \left[\alpha^\dagger \alpha, \rho_F(t) \right] + L_F[\rho_F(t)] . \tag{3.5}$$

The study of the solutions of this equation in the holomorphic representation requires the introduction of Grassmann variables θ, ξ, ..., that anticommute with the operators $\alpha^\dagger$ and α, and such that:

$$\theta\xi = -\xi\theta , \qquad \theta^2 = \xi^2 = 0 . \tag{3.6}$$

The elements $|\psi\rangle$ of the Hilbert space $\mathcal{H}_F$ are now holomorphic functions $\psi(\bar\theta)$ of the variable $\bar\theta$. However, since $\bar\theta^2 = 0$, their Taylor expansion contains only two terms: $\psi(\bar\theta) = \psi_0 + \psi_1\,\bar\theta$, with ψ_0 and ψ_1 complex parameters; they clearly represent the components of $|\psi\rangle$ along the vacuum and one-fermion states.[†]

The inner product of two states $|\phi\rangle$ and $|\psi\rangle$ involves the integration over anticommuting variables (Berezin integral), defined by the conditions $\int \theta \, d\theta = 1$ and $\int d\theta = 0$:[26]

$$\langle \phi | \psi \rangle = \int \psi^*(\theta) \, \phi(\bar\theta) \, e^{-\bar\theta\theta} \, d\bar\theta \, d\theta , \tag{3.7}$$

where $\psi^*(\theta) = \psi_0^* + \theta\psi_1^*$ is by definition the adjoint of $\psi(\bar\theta)$.

Similarly, to an operator $\mathcal{O}$ acting on $\mathcal{H}_F$ there corresponds a kernel $\mathcal{O}(\bar\theta, \theta)$; the result of its action on the vector $|\psi\rangle$ is given by:

$$\phi(\bar\theta) = \int \mathcal{O}(\bar\theta, \xi) \, \psi(\bar\xi) \, e^{-\bar\xi\xi} \, d\bar\xi \, d\xi . \tag{3.8}$$

Note that the identity operator is represented by the kernel $e^{\bar\theta\theta}$. Furthermore, in this framework the fermionic creation and annihilation operators are realized by left multiplication and differentiation with respect to $\bar\theta$:

$$\alpha^\dagger \to \bar\theta , \qquad \alpha \to \frac{\partial}{\partial\bar\theta} , \tag{3.9}$$

so that $\alpha^\dagger$ is indeed the adjoint of α with respect to the inner product in (3.7).

[†] Since the fermionic oscillator is a two-level system, a simple correspondence between the holomorphic and the standard matrix representation can easily be established; however, working with the holomorphic representation is in general more convenient, since explicit, closed expressions for ρ_F can always be given, even in presence of n degrees of freedom. See also the discussion in Sect.5

As in the bosonic case, since $\mathcal{L}_F$ in (3.5) is quadratic in the operators (3.9), the kernel $\rho_F(\bar{\theta}, \theta; t)$ representing the state ρ_F of the system can be taken to be of Gaussian form:

$$\rho_F(\bar{\theta}, \theta; t) = \gamma(t) \, e^{-\bar{\theta}\, \Gamma(t)\, \theta} \; . \tag{3.10}$$

For simplicity, also in this case we assume $\langle a^\dagger \rangle = \langle a \rangle = 0$ for all times, so that terms linear in $\bar{\theta}$ and θ are absent in (3.10). A more general Ansatz for $\rho_F(\bar{\theta}, \theta; t)$ is discussed in the Appendix. The normalization condition:

$$\mathrm{Tr}\big[\rho_F(t)\big] = \int \rho_F(\bar{\theta}, \theta; t) \, e^{\bar{\theta}\theta} \, d\theta \, d\bar{\theta} = 1 \; , \tag{3.11}$$

readily implies: $\gamma(t) = \big[1 - \Gamma(t)\big]^{-1}$, so that the kernel ρ_F in (3.10) contains only one independent unknown function. It can be conveniently recast in the following form:

$$\rho_F(\bar{\theta}, \theta; t) = \gamma(t) + \big[1 - \gamma(t)\big] \bar{\theta}\theta \; , \tag{3.12}$$

explicitly showing that γ and $1 - \gamma$ represent the two eigenvalues of ρ_F.[†] Finally, the physical meaning of $\gamma(t)$ can easily be derived:

$$\langle a a^\dagger \rangle(t) \equiv \mathrm{Tr}\big[a a^\dagger \, \rho_F(t)\big] = \int \frac{\partial}{\partial\bar{\theta}} \Big[\bar{\theta} \, \rho_F(\bar{\theta}, \theta; t)\Big] \, e^{\bar{\theta}\theta} \, d\theta \, d\bar{\theta} = \gamma(t) \; . \tag{3.13}$$

Insertion of (3.12) in the evolution equation (3.5) allows deriving the equation satisfied by the unknown function $\gamma(t)$:

$$\dot{\gamma}(t) = -2\left(\eta_F + \sigma_F\right)\gamma(t) + 2\eta_F \; , \tag{3.14}$$

whose general solution is simply:

$$\gamma(t) = e^{-2(\eta_F + \sigma_F)t}(\gamma_0 - \gamma_\infty) + \gamma_\infty \; , \tag{3.15}$$

where $\gamma_0 = \gamma(0)$ is the initial condition, while

$$\gamma_\infty = \frac{\eta_F}{\eta_F + \sigma_F} \; . \tag{3.16}$$

Since η_F and σ_F are positive constants, both $\gamma(t)$ and $1 - \gamma(t)$ are non negative, so that $0 \le \gamma(t) \le 1$. Furthermore, independently from the initial condition, the density matrix ρ_F describing the state of the fermionic oscillator always approaches for large times the equilibrium configuration: $\rho_F^\infty = \gamma_\infty + (1 - \gamma_\infty)\bar{\theta}\theta$; this is a thermal state, provided the inverse temperature β is introduced via the relation: $e^{\beta\omega_F} = \eta_F/\sigma_F$, with $\eta_F \ge \sigma_F$.

[†] Note that this simple rewriting of the Gaussian Ansatz is possible only in one dimension; in presence of n degrees of freedom, the covariance Γ would be an $n \times n$ hermitian matrix and the Taylor expansion of (3.10) would be much more involved.

The evolution towards equilibrium is not in general associated with a monotonic increase of the von Neumann entropy $S[\rho_F] \equiv S_F$. Its explicit expression can be computed using the definition (2.26):

$$S_F(t) = -\gamma(t) \ln \gamma(t) - [1 - \gamma(t)] \ln [1 - \gamma(t)] , \qquad (3.17)$$

while its time derivative reads: $\dot{S}_F = \dot{\gamma}[\ln(1 - \gamma) - \ln \gamma]$; one can easily check using (3.14) and (3.15) that $\dot{S}_F$ is always negative when γ lays between $1/2$ and γ_∞, while it is positive outside this interval.

More precisely, as a function of γ, S_F grows from its minimum value $S_F = 0$ at $\gamma = 0$ up to its maximum $S_F = \ln 2$ reached for $\gamma = 1/2$, and then decreases, becoming again zero at $\gamma = 1$. Therefore, S_F monotonically grows only when $\gamma(t)$ increases in the interval $[0, 1/2]$, or decreases in the interval $[1/2, 1]$. For $\eta_F < \sigma_F$, this happens when $\gamma_0 \leq \gamma_\infty$; indeed, this implies $\gamma(t) \leq \gamma_\infty < 1/2$ and $\dot{\gamma}(t) \geq 0$ for all times. Similar conditions hold when $\eta_F > \sigma_F$; in this case to obtain a monotonic increase of entropy, one has to choose $\gamma_0 \geq \gamma_\infty$, so that $\gamma(t) \geq \gamma_\infty > 1/2$ and $\dot{\gamma}(t) \leq 0$ for all t.

The case $\eta_F = \sigma_F$ is somehow special, since now $\dot{S}_F \geq 0$ independently from the choice of the initial state; the density matrix ρ_F asymptotically approaches the infinite-temperature, totally disordered state $\rho_F^\infty = e^{\bar{\theta}\theta}/2$, for which the entropy is maximal, $S_F = \ln 2$.

As a final remark, note that in the case of the fermionic oscillator the sufficient condition for entropy increase discussed at the end of the previous section does not lead in general to useful constraints. Indeed, the inequality in (2.32) gives now the condition:

$$\dot{S}_F(t) \geq 2(\eta_F - \sigma_F)[1 - 2\gamma(t)] . \qquad (3.18)$$

Unless $\eta_F = \sigma_F$, the r.h.s. of this inequality becomes always negative for large enough times, as it can be easily realized by substituting for $\gamma(t)$ its asymptotic value γ_∞.

4. THE SUPERSYMMETRIC OSCILLATOR

We shall now discuss the behaviour of an oscillator composed by both bosonic and fermionic degrees of freedom in interaction with an environment, under the hypothesis that its evolution is described by a quantum dynamical semigroup. The density matrix ρ representing the state of the system is now an operator on the Hilbert space $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$. Its time evolution is described by an equation of the form (2.1), where both the total hamiltonian H and the dissipative piece $L[\rho]$ are expressed in terms of bosonic, $a^\dagger$, a, and fermionic, $\alpha^\dagger$, α, creation and annihilation operators, obeying

$$[a^\dagger, \alpha^\dagger] = [a^\dagger, \alpha] = [a, \alpha^\dagger] = [a, \alpha] = 0 , \qquad (4.1)$$

together with the standard commutation, anticommutation relations.

The hamiltonian $H = H_B + H_F$, the sum of the bosonic and fermionic terms of the form (2.9) and (3.2), possesses an additional property when the two frequencies are equal: $\omega_B = \omega_F = \omega$. Indeed, the following charges:

$$Q_+ = \omega^{1/2}\, a\, \alpha^\dagger \, , \qquad Q_- = \omega^{1/2}\, a^\dagger \alpha \, , \tag{4.2}$$

commute with the hamiltonian $H = \omega(a^\dagger a + \alpha^\dagger \alpha)$, and further:

$$\{Q_+, Q_-\} = H \, , \qquad Q_+^2 = Q_-^2 = 0 \, . \tag{4.3}$$

This is the simplest example of a supersymmetry algebra. The system described by H is therefore supersymmetric and the conserved supercharges Q_+ and Q_- exchange bosons and fermions; further, from the algebra (4.3) one deduces that the ground state of H is a zero energy singlet and that all excited states form degenerate doublets.

The additional piece $L[\rho]$ in the evolution equation (2.1) will be taken to be the sum of the bosonic $L_B[\rho]$ and fermionic $L_F[\rho]$ linear operators already introduced in the previous sections. This is a natural choice since it assures integrability of the time evolution ($L[\rho]$ is again at most quadratic in the creation and annihilation operators), while avoiding mixings between bosonic and fermionic degrees of freedom induced by the dissipative term; in other terms, $L[\rho]$ is thus bosonic in character.

Nevertheless, this simple form of $L[\rho]$ does not in general assure supersymmetry invariance. In ordinary Quantum Mechanics, to an invariance of the hamiltonian there always correspond a conservation law and viceversa. For time evolution generated by equations of the form (2.1) this is usually not true: charge conservation and invariance (or symmetry) give rise to two different and in general unrelated conditions.

To further elaborate on this point, notice that to the evolution equation (2.1) for the density matrix ρ there corresponds an analogous evolution for any operator X representing an observable of the system:

$$\frac{\partial}{\partial t} X = \mathcal{L}^*[X] \equiv i[H, X] + L^*[X] \, , \tag{4.4}$$

where the linear operator L^* is the "dual" of L and it is defined via the following identity

$$\mathrm{Tr}\big(L^*[X]\, \rho\big) \equiv \mathrm{Tr}\big(X\, L[\rho]\big) \, . \tag{4.5}$$

Consider now a symmetry of the hamiltonian H generated by the charge G, inducing the following transformation on the observables:

$$X \to X' = U^{-1}\, X\, U \, , \qquad U = e^{i\,G} \, . \tag{4.6}$$

This transformation will be an invariance of the system only when it is compatible with the evolution equation (4.4), *i.e.* $\mathcal{L}^*[X'] = U^{-1}\, \mathcal{L}^*[X]\, U$, for any X; equivalently, in infinitesimal form:

$$[G, L^*[X]] = L^*\big[[G, X]\big] \, . \tag{4.7}$$

This condition is clearly distinct from the relation that guarantees the time conservation of the mean value $\langle G \rangle \equiv \text{Tr}[G\rho(t)]$ of the generator G; recalling (2.1) and (4.5), from the condition $d/dt\langle G \rangle = 0$ for any state, one readily derives:

$$L^*[G] = 0 . \tag{4.8}$$

In the case of the supersymmetric oscillator, the dual map $L^*[X] = L_B^*[X] + L_F^*[X]$ can be easily deduced from (2.11) and (3.4). Explicitly, one finds:

$$
\begin{aligned}
L_B^*[X] &= \eta_B\big(2\,a^\dagger X a - X a^\dagger a - a^\dagger a X\big) + \sigma_B\big(2\,a X a^\dagger - X a a^\dagger - a a^\dagger X\big) \\
&\quad + \lambda_B\big(2\,a^\dagger X a^\dagger - X a^{\dagger 2} - a^{\dagger 2} X\big) + \lambda_B^*\big(2\,a X a - X a^2 - a^2 X\big) , \\
L_F^*[X] &= \eta_F\big(2\,\alpha^\dagger X \alpha - X \alpha^\dagger \alpha - \alpha^\dagger \alpha X\big) + \sigma_F\big(2\,\alpha X \alpha^\dagger - X \alpha \alpha^\dagger - \alpha \alpha^\dagger X\big) \\
&\quad + 2\,\lambda_F\,\alpha^\dagger X \alpha^\dagger + 2\,\lambda_F^*\,\alpha X \alpha .
\end{aligned}
\tag{4.9}
$$

The parameter of a supersymmetry transformation is anticommuting, so that the corresponding generator takes the form $G = \xi\,Q_+$, where ξ is a Grassmann variable, commuting with bosonic operators, but anticommuting with the fermionic ones. Inserting it in (4.7) and using (4.9), after some algebraic manipulations one gets the following condition:

$$\big(\eta_B - \sigma_B - \eta_F + \sigma_F\big)[X, G] + \lambda_F^*\{X, G^\dagger\} = 0 . \tag{4.10}$$

Since this relation must be true for any observable X, supersymmetry invariance is compatible with the time evolution only when:

$$\eta_B - \sigma_B = \eta_F - \sigma_F , \qquad \lambda_F = 0 . \tag{4.11}$$

The holomorphic representation is again particularly useful in order to discuss the behaviour of the state $\rho(t)$ of the supersymmetric oscillator. The elements of the Hilbert space $\mathcal{H}$ will be now represented by holomorphic functions of the complex variable $\bar{z}$ and of the Grassmann symbol $\bar{\theta}$, while creation and annihilation operators will act on them following the rules in (2.16) and (3.9). The density matrix ρ will be now a kernel $\rho(\bar{z}, z; \bar{\theta}, \theta)$, whose explicit expression can be taken to be of Gaussian form.[†] It can be expanded as:

$$\rho(\bar{z}, z; \bar{\theta}, \theta) = \rho_0(\bar{z}, z) + \rho_1(\bar{z}, z)\,\bar{\theta}\theta . \tag{4.12}$$

The normalization condition $\text{Tr}[\rho] = 1$ now involves both ordinary and Grassmann integrals:

$$\int \rho(\bar{z}, z; \bar{\theta}, \theta)\,e^{-\bar{z}z}\,e^{\theta\bar{\theta}}\,d\bar{z}\,dz\,d\theta\,d\bar{\theta} = \int [\rho_0(\bar{z}, z) + \rho_1(\bar{z}, z)]\,e^{-\bar{z}z}\,d\bar{z}\,dz = 1 . \tag{4.13}$$

Inserting the Ansatz (4.12) into the evolution equation for ρ allows deriving the following conditions on the bosonic kernels ρ_0 and ρ_1:

$$\dot{\rho}_0(t) = \mathcal{L}_B[\rho_0(t)] + 2\big[\eta_F\,\rho_1(t) - \sigma_F\,\rho_0(t)\big] , \tag{4.14a}$$

$$\dot{\rho}_1(t) = \mathcal{L}_B[\rho_1(t)] - 2\big[\eta_F\,\rho_1(t) - \sigma_F\,\rho_0(t)\big] , \tag{4.14b}$$

[†] Here again we assume vanishing initial averages $\langle a^\dagger \rangle$, $\langle a \rangle$, $\langle \alpha^\dagger \rangle$, $\langle \alpha \rangle$.

where the linear operator $\mathcal{L}_B[\rho]$ is as in (2.10). It follows that the combination $\rho_0 + \rho_1$ satisfies the same evolution equation discussed in Sect.2 for the case of a single bosonic oscillator. The Gaussian Ansatz $\rho_B(\bar{z}, z)$ in (2.17) can equally well be adopted here for $\rho_0 + \rho_1$, since performing the Grassmann integrations, one consistently finds (compare with (2.20)):

$$\langle a^2 \rangle(t) \equiv \mathrm{Tr}\big[a^2\, \rho(t)\big] = \int \frac{\partial^2}{\partial \bar{z}^2}\big[\rho_0(\bar{z}, z; t) + \rho_1(\bar{z}, z; t)\big]\ e^{-\bar{z}z}\ d\bar{z}\, dz = x(t)\ ,$$

$$\langle a^{\dagger 2} \rangle(t) \equiv \mathrm{Tr}\big[a^{\dagger 2}\, \rho(t)\big] = \int \bar{z}^2\big[\rho_0(\bar{z}, z; t) + \rho_1(\bar{z}, z; t)\big]\ e^{-\bar{z}z}\ d\bar{z}\, dz = \bar{x}(t)\ , \qquad (4.15)$$

$$\langle a\, a^\dagger \rangle(t) \equiv \mathrm{Tr}\big[a\, a^\dagger\, \rho(t)\big] = \int \frac{\partial}{\partial \bar{z}}\bar{z}\big[\rho_0(\bar{z}, z; t) + \rho_1(\bar{z}, z; t)\big]\ e^{-\bar{z}z}\ d\bar{z}\, dz = y(t)\ .$$

As a consequence, the time evolution of these quantities is that given in (2.22) and (2.24).
Inserting back this result into (4.14a), one obtains:

$$\dot{\rho}_0(t) = \mathcal{L}_B[\rho_0(t)] - 2\left(\eta_F + \sigma_F\right)\rho_0(t) + 2\eta_F\, \rho_B(t)\ . \qquad (4.16)$$

The form of this equation suggests to look for a solution in which $\rho_0(t)$ differs from $\rho_B(t)$ by an unknown multiplicative function $\gamma_F(t)$. It can be identified with the function $\gamma(t)$ studied in the previous section, since it satisfies the same equation (3.14) and has the same physical meaning:

$$\langle \alpha\alpha^\dagger \rangle(t) \equiv \mathrm{Tr}\big[\alpha\alpha^\dagger\, \rho(t)\big] = \int \rho_0(\bar{z}, z; t)\ e^{-z\bar{z}}\ dz\, d\bar{z} = \gamma_F(t)\ . \qquad (4.17)$$

As a consequence, $\rho_1 = (1 - \gamma_F)\rho_B$, and therefore one finally finds:

$$\rho(\bar{z}, z; \bar{\theta}, \theta) = \Big[\gamma_F + (1 - \gamma_F)\, \theta\bar{\theta}\Big]\rho_B(\bar{z}, z) \equiv \rho_F(\bar{\theta}, \theta)\ \rho_B(\bar{z}, z)\ . \qquad (4.18)$$

Not surprisingly, the density matrix that solves the evolution equation (2.1) in the case of the supersymmetric oscillator is in factorized form; its behaviour can be deduced from the analysis of the previous sections, provided the conditions (4.11) for supersymmetry invariance are taken into account.

In particular, ρ approaches an equilibrium state for large times only when $\eta_B > \sigma_B$, which also implies: $\eta_F > \sigma_F$. This limiting state is thermal, with inverse temperature β, only for $\lambda_B = 0$ and $\eta_B/\sigma_B = \eta_F/\sigma_F = e^{\beta\omega}$, which implies, recalling the condition (4.11): $\eta_B = \eta_F$ and $\sigma_B = \sigma_F$.[30]

Also in the case of the supersymmetric oscillator, the total entropy does not have in general a monotonic behaviour during the approach to equilibrium. Since the density matrix ρ is in factorized form, the total entropy S will be the sum of the bosonic and fermionic contributions. Using the variable $\gamma_B = \chi - 1/2 \geq 0$, where χ is defined as in (2.30), and recalling the results of the previous sections, one has:

$$S = \left(\gamma_B + 1\right)\ln\left(\gamma_B + 1\right) - \gamma_B \ln \gamma_B - \left(\gamma_F - 1\right)\ln\left(\gamma_F - 1\right) - \gamma_F \ln \gamma_F\ . \qquad (4.19)$$

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Its time derivative, that can be expressed as:

$$\dot{S} = \dot{\gamma}_B \ln\left(1 + \frac{1}{\gamma_B}\right) + \dot{\gamma}_F \ln\left(1 - \frac{1}{\gamma_F}\right) \,, \tag{4.20}$$

does not have in general a definite sign, although possible compensations between the bosonic and fermionic contributions can concur to a positive r.h.s. for certain time intervals.

As discussed at the end of Sect.2, a bound on $\dot{S}$ can be obtained by working directly with the definition (2.26) and the equation (2.1). In the present case, this procedure gives:

$$\dot{S} \geq 2(\sigma_B - \eta_B) + 2(\sigma_F - \eta_F)[2\gamma_F - 1] \equiv 4(\sigma_F - \eta_F)\gamma_F \,, \tag{4.21}$$

where the identity is a consequence of the condition (4.11). Since $0 \leq \gamma_F(t) \leq 1$, the inequality (4.21) assures $\dot{S} \geq 0$ for $\sigma_F \geq \eta_F$. However, this condition would lead to a rather singular behaviour for the bosonic part of the density matrix in (4.18), and thus for the whole ρ. In fact, also σ_B would be greater than η_B and, as discussed in Sect.2, this implies an infinitely growing average occupation number. In conclusion, although inducing a partial compensation between the bosonic and fermionic contributions to S, the supersymmetry condition (4.11) is in general not enough to guarantee monotonic entropy increase for all times during the evolution of the system.

5. DISCUSSION

All the considerations developed in the previous sections for single oscillators can be generalized to the case of n independent oscillators, both bosonic and fermionic. Their interaction with an external environment can still be consistently described in terms of quantum dynamical semigroups, so that their time evolution can be modelled by means of equations of the form (2.1), (2.3), with operators L_k linear in the relevant fundamental variables. However, the coefficients r and s of (2.4) become now matrices, and the number of independent constants characterizing the dissipative part $L[\rho]$ rapidly increase with n, making the evolution equation (2.1) rather involved.

Nevertheless, various simplifying conditions can be imposed to reduce, at least in part this arbitrariness. Those involving symmetry properties are the most physically interesting. As discussed in the Introduction, the interaction between system and environment can be considered in general to be weak; therefore, in many instances, the presence of the environment should not to be able to alter the symmetry properties of the system. In the case of n isotropic oscillators, the hamiltonian is invariant under the action of the group $SU(n)$; it is then quite natural to assume the same invariance property to be valid for the full evolution equation. As discussed in Sect.4, this can be achieved by imposing the condition (4.7) for any element G of the $SU(n)$ algebra.

Also in this more general setting, the holomorphic representation appears to be a particularly convenient framework to analyze the behaviour of the solutions of (2.1). It requires the introduction of n commuting or anticommuting complex symbols, that allow

realizing the corresponding creation and annihilation operators as multiplication and differentiation by these variables. The kernel representing the system density matrix can still be taken to have a generic Gaussian functional expression. However, the various "coefficients" in the exponent, suitable generalizations of the functions x, $\bar{x}$ and y in the bosonic case and of Γ in the fermionic one, become now $n \times n$ matrices. They obey quadratic (Riccati-like) time evolution equations, whose solutions can always be obtained, albeit in general in terms of implicitly defined functions.[31, 32]

Although developed in the analysis of simple open systems, the techniques described in the previous sections are actually very general; they can be used to study the dynamics of more complicated models, for which the operator $\mathcal{L}$ in (2.1) is not quadratic in the relevant variables. In these cases, complete explicit expressions for the density matrix ρ as solution of (2.1) can not in general be given. Nevertheless, approximate expressions for ρ, typically in Gaussian form, can be obtained via the application of suitable variational procedures.

Indeed, equations of the form (2.1) can be derived by mean of a suitable variational principle,[33] obtained by generalizing the one yielding the Liouville - von Neumann equation in ordinary Quantum Mechanics.[28, 34] In the case of isoentropic time evolutions, these variational techniques have allowed detailed discussions of a wide range of physical phenomena, from statistical physics to inflationary cosmology.[34, 35] Their application to the study of quantum dynamical semigroups within the framework presented in the previous sections will surely provide new insights on the behaviour of open quantum systems.

APPENDIX

The Gaussian kernels $\rho_B(\bar{z}, z; t)$ and $\rho_F(\bar{\theta}, \theta; t)$ representing the density matrices for the bosonic and fermionic oscillators discussed in Sect.2 and 3 lead to vanishing averages for the corresponding creation and annihilation operators. This condition can easily be released by introducing a more general Ansatz.

In the bosonic case, take:

$$\rho_B(\bar{z}, z; t) = \frac{1}{\sqrt{N(t)}} e^{-\frac{1}{2N(t)} \left[2\, y(t)\, [\bar{z} - \bar{v}(t)][z - v(t)] - \bar{x}(t)\, [z - v(t)]^2 - x(t)[\bar{z} - \bar{v}(t)]^2 \right] + \bar{z}z} \, , \qquad (A.1)$$

that differ from the expression (2.17) because of the presence of the two additional functions $v(t)$ and $\bar{v}(t)$. Trace conservation for all times, $\text{Tr}[\rho_B(t)] = 1$, still implies:

$$N(t) = y^2(t) - |x(t)|^2 \, , \qquad (A.2)$$

while hermiticity requires: $\bar{v}(t) \equiv [v(t)]^*$. With this choice for $\rho_B(\bar{z}, z; t)$, the averages of $a^\dagger$ and a are in general non vanishing:

$$\langle a^\dagger \rangle(t) \equiv \text{Tr}\big[a^\dagger \, \rho_B(t)\big] = \int \bar{z} \, \rho_B(\bar{z}, z; t) \; e^{-\bar{z}z} \; d\bar{z} \, dz = \bar{v}(t) \, ,$$

$$\langle a \rangle(t) \equiv \text{Tr}\big[a \, \rho_B(t)\big] = \int \frac{\partial}{\partial \bar{z}} \big[\rho_B(\bar{z}, z; t)\big] \; e^{-\bar{z}z} \; d\bar{z} \, dz = v(t) \, . \qquad (A.3)$$

The time evolution equation (2.10) implies the following homogeneous equation for $v(t)$

$$\dot{v}(t) = 2 \left(\eta_B - \sigma_B + i\omega_B \right) v(t) \, , \tag{A.4}$$

so that $v(t)$ is non vanishing only if its initial value $v(0)$ is different from zero:

$$v(t) = e^{-(\eta_B - \sigma_B + i\omega_B)t} \, v(0) \, . \tag{A.5}$$

The physical meaning of the remaining functions x, $\bar{x}$ and y appearing in $(A.1)$ is slightly changed with respect to those studied in Sect.2; now one finds:

$$\begin{aligned}
x &= \langle a^2 \rangle - \langle a \rangle^2 \, , \\
\bar{x} &= \langle a^{\dagger 2} \rangle - \langle a^\dagger \rangle^2 \, , \\
y &= \langle a\, a^\dagger \rangle - \langle a \rangle \langle a^\dagger \rangle \, .
\end{aligned} \tag{A.6}$$

Nevertheless, one can check that these functions still obey the evolution equations (2.21), so that the considerations and the discussions of Sect.2 apply to this more general situation as well.

In the case of the fermionic oscillator, the most general Gaussian Ansatz for the kernel $\rho_F(\bar{\theta}, \theta; t)$ can be written as:

$$\rho_F(\bar{\theta}, \theta; t) = \gamma(t) \, e^{-\frac{1}{\gamma(t)} \left[\bar{\theta}\, \Delta(t)\, \theta - \bar{\varphi}(t)\, \theta - \varphi(t)\, \bar{\theta} \right]} \, . \tag{A.7}$$

The normalization condition $\mathrm{Tr}\left[\rho_F(t) \right] = 1$ gives $\Delta(t) = \gamma(t) - 1$, while hermiticity implies $\bar{\varphi}(t) = [\varphi(t)]^*$.

By performing the integration over the anticommuting variables, one finds that the function $\gamma(t)$ retains its meaning as $\langle \alpha\, \alpha^\dagger \rangle$ also in this more general setting, and therefore still obeys the evolution equation (3.14).

On the other hand, the two additional functions $\bar{\varphi}(t)$ and $\varphi(t)$ in $(A.7)$ represent the averages of $\alpha^\dagger$ and α,

$$\begin{aligned}
\langle \alpha^\dagger \rangle(t) &\equiv \mathrm{Tr}[\alpha^\dagger \, \rho_F(t)] = \int \bar{\theta} \, \rho_F(\bar{\theta}, \theta; t) \, e^{\bar{\theta}\theta} \, d\theta \, d\bar{\theta} = \bar{\varphi}(t) \, , \\
\langle \alpha \rangle(t) &\equiv \mathrm{Tr}[\alpha \, \rho_F(t)] = \int \frac{\partial}{\partial \bar{\theta}} \left[\rho_F(\bar{\theta}, \theta; t) \right] e^{\bar{\theta}\theta} \, d\theta \, d\bar{\theta} = \varphi(t) \, ,
\end{aligned} \tag{A.8}$$

and, as a consequence of (3.5), obey the following evolution equations:

$$\begin{aligned}
\dot{\varphi}(t) &= -2 \left(\eta_F + \sigma_F + i\omega_F \right) \varphi(t) + 2\lambda_F \, \bar{\varphi}(t) \, , \\
\dot{\bar{\varphi}}(t) &= -2 \left(\eta_F + \sigma_F - i\omega_F \right) \bar{\varphi}(t) + 2\lambda_F^* \, \varphi(t) \, .
\end{aligned} \tag{A.9}$$

The general solution is given by:

$$\varphi(t) = e^{-(\eta_F + \sigma_F)t} \left\{ \left[\cos(\Omega_F t) - \frac{i\omega_F}{\Omega_F} \sin(\Omega_F t) \right] \varphi(0) + \frac{2\,\lambda_F}{\Omega_F} \sin(\Omega_F t) \, \bar{\varphi}(0) \right\} , \tag{A.10}$$

where $\Omega_F = \left[\omega_F^2 - 4|\lambda_F|^2 \right]^{1/2}$ for $\omega_F \geq |\lambda_F|$. Hyperbolic functions appear in the expression $(A.10)$ when $\omega_F < |\lambda_F|$; however, thanks to the inequality $|\lambda_F|^2 \leq \eta_F \sigma_F$ (compare with (2.13)), $\varphi(t)$ always vanishes for large times.

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